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An equilibrium analysis of linear, proportional and uniform allocation of scarce capacity

GÉRARD P. CACHON and MARTIN A. LARIVIERE

The Fuqua School of Business, Duke University, Box 90120, Durham, NC 27708, USA
E-mail: gpc@mail.duke.edu or larivier@mail.duke.edu

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In many industries a supplier's total demand from the retailers she supplies frequently exceeds her capacity. In these situations, the supplier must allocate her capacity in some manner. We consider three allocation schemes: proportional, linear and uniform. With either proportional or linear allocation a retailer receives less than his order whenever capacity binds. Hence, each retailer has the incentive to order strategically; retailers order more than they desire in an attempt to ensure that their ultimate allocation is close to what they truly want. Of course, they will receive too much if capacity does not bind. In the capacity allocation game, each retailer must form expectations on how much other retailers actually desire (which is uncertain) and how much each will actually order, knowing that all retailers face the same problem. We present methods to find Nash equilibria in the capacity allocation game with either proportional or linear allocation. We find that behavior in this game with either of those allocation rules can be quite unpredictable, primarily because there may not exist a Nash equilibrium. In those situations any order above one's desired quantity can be justified, no matter how large. Consequently, a retailer with a high need may be allocated less than a retailer with a low need; clearly an *ex post* inefficient allocation. However, we demonstrate that with uniform allocation there always exists a unique Nash equilibrium. Further, in that equilibrium the retailers order their desired amounts, i.e., there is no order inflation. We compare supply chain profits across the three allocation schemes.

1. Introduction

When demand is uncertain and capacity is costly, a supplier will not build an amount of capacity sufficient to cover every possible demand realization. As a result, on occasion retailers (the supplier's customers) will demand more than she can deliver. In such a setting, the supplier must employ a mechanism to allocate the available capacity among the retailers. Proportional allocation is perhaps the most intuitive scheme for dividing capacity: If a retailer's order is $x\%$ of total orders, he receives $x\%$ of available capacity. Other schemes exist. With linear allocation, the difference between total orders and capacity is divided by the number of retailers, and this amount is subtracted from each order. (If a negative allocation results for someone, that retailer receives a zero allocation and the process is repeated with the remaining retailers). Both linear and proportional allocation ensure that every retailer receives less than his order when capacity is allocated. Further, with each of these schemes a retailer can ensure a larger allocation if he orders a larger quantity. Since retailers realize that allocation is a possibility, they all have an incentive to inflate their orders above their desired allocation. Hence, both linear and proportional allocation are "order-inflating" mechanisms. In contrast, uniform allocation is a "truth-inducing" mechanism, i.e., under uniform allocation the retailers order only their desired amounts, no matter the behavior of the other retailers or the amount of available capacity. With uniform allocation the supplier equally divides the available capacity among the retailers; if any retailer orders less than his equal share, he receives his order and the remaining capacity is allocated equally among the remaining retailers.

Allocation mechanisms have been employed in industries ranging from automobiles [1], to pharmaceuticals [2], to toys [3] but the issue has received little formal study. Lee *et al.* [4] recognize that proportional allocation creates an incentive for retailers to raise their orders above their desired allocation. However, they do not determine by how much the retailers will inflate their orders or if order inflation is at all predictable (i.e., if there exists a Nash equilibrium in order quantities). Cachon and Lariviere [5] delve deeper into the issue of capacity allocation. They identify several different allocation schemes and demonstrate that some allocation schemes induce retailers to inflate their orders whereas others do not. Since linear and proportional allocation create complex behavior, they assume the supplier implements relaxed linear allocation. That scheme, while computationally

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tractable, allows a retailer to "receive" a negative allocation, implying that it cannot be implemented in practice. The allocation schemes we consider can be implemented in practice.

In our model there is one supplier and two retailers. Retailers face one of two possible downward sloping linear demand curves. Each retailer knows his own demand and thus his desired allocation. However neither retailer knows the other's demand. Hence, the retailers are not necessarily sure whether capacity will bind and may inflate their orders above their ideal allocations.

We show that behavior in this game is quite complex with either linear or proportional allocation. For many capacity levels, there does not exist a pure strategy Nash equilibrium in order quantities. In these settings a retailer can rationalize any order above his ideal amount, no matter how large. There can also exist the possibility of multiple Nash equilibria; the players consequently cannot be sure that they are playing the same equilibrium. For linear allocation, we show that for any set of model parameters there exists a range of capacity values over which multiple equilibria are feasible. While it is possible to explicitly evaluate all equilibria with linear allocation (if they exist), with proportional allocation it is necessary to conduct some numerical search to find equilibria. Unfortunately it cannot be guaranteed that a numerical search will find all (or even any) of the equilibria with proportional allocation.

These games are analytically complex primarily because a player's profit is not necessarily unimodal in his *order*, even though a player's profit is unimodal in his *allocation*. Hence, a player's best reply correspondence, the set of optimal orders given the behavior of the other retailer, is not necessarily convex. For example, it may be optimal to order either 8 or 12 while an order of 10 yields sub-optimal profits in expectation. Convexity of the best reply correspondence is required by the standard proofs for the existence of a pure strategy equilibrium [6,7].

There is another problem. Existence theorems also require convex strategies; we cannot allow the players to choose an infinitely high order although it is rather artificial to suppose that the supplier places an *a priori* limit on the size of retailer orders. Without this limit, however, the capacity allocation game can degenerate into "who can name the largest number," which clearly has no equilibrium. (No matter what order you submit, I can always find a larger order). In a numerical study we find that the lack of a Nash equilibrium is common with either linear or proportional allocation, especially as capacity becomes very tight. We also find that while multiple equilibria may exist under either scheme, this appears to be a less common problem.

Behavior with uniform allocation is sharply different: there always exists a unique Nash equilibrium with each retailer ordering his ideal allocation, i.e., there is no order inflation. We compare supply chain profits with uniform allocation to supply chain profits with either linear or proportional allocation. We show that when there exists an equilibrium with either proportional or linear allocation, supply chain profits with those schemes are generally somewhat higher than with uniform allocation. This occurs because order inflation increases the supplier's profits, i.e., due to order inflation the supplier expects to sell a greater fraction of her capacity. Further, under any of those schemes the allocation of capacity across the retailers is reasonable, i.e., the highest need retailer generally receives the highest allocation. We conclude that when there exists a linear or proportional allocation equilibrium, order inflation does not create tremendous supply chain inefficiency, and may even benefit the supply chain.

However, our conclusion is different when there does not exist an equilibrium with either of the inflation-inducing allocation schemes. In those situations we expect rampant order inflation to lead to arbitrary allocations: since his opponent can justify any order (no matter how large) a retailer might receive nearly all of capacity or a very small allocation for any fixed order. The retailers' profits are thus significantly lower than what they would be with uniform allocation, but since the retailers' expected orders with order inflation are higher than their orders with uniform allocation, the supplier's profits are significantly higher. Whether the supply chain is better off with order inflation relative to uniform allocation depends on the how the profits are allocated in the chain. The supplier holds most of the profits when the wholesale price is relatively high, in which case order inflation benefits the supply chain. However, with a relatively low wholesale price most of the chain's profits are at the retail level. In that case order inflation significantly reduces supply chain profits.

The next section details the model. Section 3 outlines a method to find equilibria when the supplier implements linear allocation and Section 4 discusses the search for equilibria with proportional allocation. Section 5 analyzes uniform allocation. Section 6 presents the numerical study. Section 7 discusses the results and suggests possible extensions to the model. The last section concludes.

2. Model

This section explains the rules of the capacity allocation game. There are two players. Each is a retailer of a single good produced by one upstream supplier. A retailer can be one of two types. A "high" type retailer expects greater demand for the good than a "low" type retailer. Specifically, a type $t \in \{l, h\}$ retailer faces a downward sloping linear demand curve,

$$p_t = z_t - q_t.$$

where q_t is the number of units the retailer sells, p_t is the price per unit the retailer receives, and $z_l = \alpha \in (0, 1)$, $z_h = 1$. This model of demand is quite reasonable: a retailer can increase sales only by lowering its price, and a retailer with high demand expects to sell more at any given price than a low demand retailer.

The retailers' demands are independent, i.e., they are local monopolists. For example, the retailers may be located in significantly distant geographic regions. Hence, the retailers do not compete against each other in the consumer market. (Cournot and Bertrand competition are two well known forms of competition between retailers in the consumer market, so neither of them apply in this model). Instead, the retailers will only compete for scarce capacity through their orders.

Each retailer knows his own type but does not know for certain the other's type. However, each knows that the other retailer is a high type with probability $\rho \in (0, 1)$ and a low type with probability $(1 - \rho)$. We assume that retailer types are independent. Thus, whether retailer i is a high or a low type, he expects retailer j to be a high type with probability ρ.

In the game's only move, each retailer submits an order to the supplier. The supplier initiates production only after receiving the orders. She can produce up to K units, and the marginal cost of production is normalized to zero. The supplier sells units to the retailers at a constant per unit price w. If total orders are less than K, the supplier produces just enough to fill the orders. However, if orders exceed K, the supplier produces K and implements an allocation rule to divide her production between the retailers. Once the allocation is announced, the retailers must purchase their full allocation and may not return goods to the supplier. However, the supplier cannot allocate more to a retailer than he has ordered. Three allocation mechanisms are considered, linear, proportional and uniform allocation. (The details of these allocation mechanisms are explained later).

Let $\pi_t(a)$ be a type t retailer's profit when he is allocated a units,

$$\pi_t(a) = (z_t - a - w)a.$$

where it is assumed that the retailer must sell all of the units he has been allocated. The retail price is then negative when $a > z_t$. In practice, a retailer should withhold stock from the market when a is greater than $z_t/2$, the revenue maximizing sales level. Introducing that assumption has little theoretical value but does add an additional layer of computational effort. (Whether a retailer can withhold stock or not, a retailer's profits are strictly concave in his allocation, so that assumption has no qualitative impact on the results).

Define a player's strategy as a set of order quantities, one for each type. Let $X = \{x_l, x_h\}$ be retailer i's strategy, where $x_t \geq 0$ is the amount retailer i orders when he is type t. Similarly, let $Y = \{y_l, y_h\}$ be retailer j's strategy. (It is convenient to assume that a player chooses an order quantity for each of his possible types even though he will know his type before submitting his order). A pure strategy Nash equilibrium is a pair of strategies such that neither player ever has a profitable unilateral deviation. All parameters of the game (outside of the retailers' types) are common knowledge. For example, the retailers know the supplier's capacity and the allocation mechanism that may be implemented. Players are risk neutral so they choose their strategies to maximize their expected profits.

A type t retailer's ideal allocation is $a_t = (z_t - w)/2$. Assume $w < \alpha$, so even a low type desires a positive allocation. In this game each retailer's challenge is to obtain an allocation as close to a_t as possible.

3. Linear allocation

When the supplier implements linear allocation a retailer ordering x is allocated $a(x, y)$ when the other retailer orders y, where

$$a(x, y) = \begin{cases} x & x + y \leq K, \\ (x - y + K)/2 & x + y \geq K, \quad |x - y| < K, \\ K & x + y \geq K, \quad x - y > K, \\ 0 & x + y \geq K, \quad y - x > K. \end{cases}$$

In words, a retailer receives his order whenever the sum of all orders is less than capacity, K. When orders exceeds capacity, half of the difference between total orders and capacity is deducted from each order, assuming the deduction is less than the smallest retailer order. If the deduction is greater than the smallest retailer order, that retailer receives a zero allocation and the other retailer receives all of the capacity.

It is straight forward to show that if the retailers were to report their types truthfully, linear allocation is the only allocation mechanism that would maximize total retailer profits. This is clearly a desirable property, and it is our primary motivation for studying linear allocation.

3.1. *Reaction correspondence*

Suppose retailer i expects that Y will be retailer j's strategy. Define $r_t(Y)$ as retailer i's reaction correspondence, i.e., the set of orders that maximizes retailer i's profits when his type is t, $\Pi_t(x_t, Y)$,

$$\Pi_t(x_t, Y) = (1 - \rho)\pi_t(a(x_t, y_l)) + \rho\pi_t(a(x_t, y_h)).$$

The first step in the analysis of $r_t(Y)$ is to restrict retailer i's expectations of retailer j's strategy. While there are no *a priori* restrictions on retailer i's expectation, for the purpose of equilibrium analysis there are clearly some expectations that are unreasonable. For example, it is unreasonable to expect that a retailer will order less than his optimal quantity or that a high type will order less than a low type. (All proofs are in Appendix A).

Lemma 1. *In any equilibrium* $a_t \le x_t$ *and* $a_t \le y_t$. *Further,* $x_l \le x_h$ *and* $y_l \le y_h$.

There are several situations in which it is straightforward to determine retailer i's optimal order. First, suppose $a_t + y_h \le K$. Retailer i can expect to receive his desired allocation, regardless of retailer j's type; he should simply order a_t. Second, suppose capacity is quite restrictive, $K \le a_t$. Retailer i wants all of the capacity since even that quantity is less than he desires. The retailer can achieve that allocation with an order no less than $K + y_h$. (This, of course, would mean that the other retailer would receive a zero allocation).

Finding a retailer's optimal order quantity for intermediate capacities, $a_t < K < a_t + y_h$, is more complex because his profit function is not necessarily (or even likely) unimodal in his order quantity. A solution to this problem is to divide a retailer's profit function into multiple intervals such that the profit function is unimodal within each interval. Locally optimal orders for each interval are compared to yield the set of globally optimal orders.

Consider retailer i's marginal profit if he knew that retailer j would order y,

$$\frac{\partial \pi_t(a(x_t, y))}{\partial x_t} = \begin{cases} 0 & x_t < y - K, \\ z_t - w - 2x_t & y - K \le x_t < y + K,\ x_t < K - y, \\ (z_t - w - x_t + y - K)/2 & y - K \le x_t < y + K,\ K - y \le x_t, \\ 0 & y + K \le x_t. \end{cases}$$

In the first case, retailer i receive a zero allocation. In the second, he is allocated his order. In the third, he is allocated less than his order but less than K. In the final case, he is allocated all of the capacity.

Of course, retailer i does not know retailer j's order for certain. Hence, depending on whether retailer j is a high or low type, retailer i faces the possibility of the eight different *allocation scenarios* that are listed in Table 1.

Table 1. The potential scenarios

Scenario	*Retailer j is a low type*	*Retailer j is a high type*
1	$a(x_t, y_l) = x_t$	$0 < a(x_t, y_h) < x_t$
2	$0 < a(x_t, y_l) < x_t$	$0 < a(x_t, y_h) < x_t$
3	$a(x_t, y_l) = K$	$0 < a(x_t, y_h) < x_t$
4	$0 < a(x_t, y_l) < x_t$	$a(x_t, y_h) = 0$
5	$a(x_t, y_l) = x_t$	$a(x_t, y_h) = 0$
6	$a(x_t, y_l) = 0$	$a(x_t, y_h) = 0$
7	$a(x_t, y_l) = K$	$a(x_t, y_h) = 0$
8	$a(x_t, y_l) = K$	$a(x_t, y_h) = K$

For each scenario, we can determine retailer i's expected marginal profit:

$$\Pi_t' = \begin{cases} (2(1-\rho)(\phi_t - x_t) + \rho(\phi_t + y_h - K))/2 & y_h - K \le x_t < K - y_l, \\ ((1-\rho)(\phi_t + y_l - K) + \rho(\phi_t + y_h - K))/2 & \max\{K - y_l, y_h - K\} \le x_t < K + y_l, \\ (\rho(\phi_t + y_h - K))/2 & \max\{y_l + K, y_h - K\} \le x_t < y_h + K, \\ ((1-\rho)(\phi_t + y_l - K))/2 & \max\{y_l - K, K - y_l\} \le x_t < \min\{y_l + K, y_h - K\}, \\ (1-\rho)(\phi_t - x_t) & y_l - K \le x_t < \min\{K - y_l, y_h - K\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\phi_t = z_t - w - x_t$ and $x_t \ge a_t$ is assumed throughout. The first five cases correspond to the first five allocation scenarios listed above while the last case corresponds to the last three allocation scenarios. For the first five cases, second order conditions confirm the profit function is strictly concave in retailer i's order. Thus retailer i's best order within each interval either satisfies the first order condition or equals an interval boundary. Let $r_t^1(Y)$, $r_t^2(Y)$, $r_t^3(Y)$ and $r_t^4(Y)$ be the solutions to the first order conditions in the first four cases in (1), ignoring the boundaries of the intervals,

$$r_t^1(Y) = \frac{2(z_t - w) - \rho(z_t + K - w - y_h)}{4 - 3\rho},$$

$$r_t^2(Y) = z_t - w - K + y_l + \rho(y_h - y_l),$$

$$r_t^3(Y) = z_t - w - K + y_h,$$

$$r_t^4(Y) = z_t - w - K + y_l.$$

The next theorem, indicates that in the intermediate capacities an optimal order must satisfy one of the above first order conditions, i.e., none of the interval boundaries are optimal.

Theorem 2. *In the capacity allocation game*

$$r_t(Y) = \begin{cases} a_t & a_t + y_h \le K, \\ \{x \in r_t^1(Y), r_t^2(Y), r_t^3(Y), r_t^4\{Y\} : \\ \quad x = \arg\max_{x_t} \Pi_t(x_t, Y)\} & a_t < K < a_t + y_h, \\ \{x : x \ge K + y_h\} & K \le a_t. \end{cases}$$

So for intermediate capacity levels the set of optimal orders is a subset of $\{r_t^1(Y), r_t^2(Y), r_t^3(Y), r_t^4\{Y\}\}$. Since this is not a convex set, the set of optimal orders is not convex.

3.2. *Equilibria*

This section identifies pure strategy Nash equilibria in the capacity allocation game with linear allocation. There are some cases in which this task is simple. Suppose capacity is ample, $K > 2a_h$. Each retailer could order a_t and be assured of receiving a_t. A retailer could never do better, so this is the unique Nash equilibrium.

Suppose capacity is quite tight, $K < a_h$. Neither high type retailer is ever satisfied. To secure the full capacity, each high type will try to order K more than the other high type retailer. Hence, the game reduces to "who can name the largest number," and naturally there is no Nash equilibrium. The absence of an equilibrium is also easy to identify at a possibly higher capacity level. Suppose $K < 2a_l$. In that case, the low type retailer with the smaller order always receives less than a_l, and so always has an incentive to raise his order, thereby destroying any possible equilibrium.

The remainder of this section assumes that capacity is at an intermediate level, i.e., $K \in (\max\{2a_l, a_h\}, 2a_h)$. The search for equilibria begins by eliminating some orders from consideration. The next lemma implies that in equilibrium each retailer type must expect to receive a positive allocation.

Lemma 3. *There does not exist a Nash equilibrium in which* $x_h \geq K + y_l$, *or* $y_h \geq K + x_l$.

The previous lemma is used to obtain the next result.

Theorem 4. *In any Nash equilibrium* $x_t \in \{r_t^1(Y), r_t^2(Y)\}$.

From Theorem 4, it follows that in equilibrium some retailer must expect to receive his order for some realization of types.

Lemma 5. *There does not exist a Nash equilibrium in which every type expects capacity to bind always.*

So what remains as a possible Nash equilibrium? We focus on symmetric equilibria, i.e., $X = Y$. (Asymmetric equilibria are discussed in Appendix B). Two types of symmetric equilibria are possible.

In a Type 1 equilibrium capacity binds only when there are two high type retailers. Let $X^* = \{x_l^*, x_h^*\}$ be this equilibrium. To be consistent with expectations, it must follow that $2x_h^* > K$ and $x_l^* + x_h^* \leq K$. Consider the following candidate equilibrium

$$x_l^* = a_l, \quad x_h^* = \frac{(2-\rho)(1-w) - \rho K}{4(1-\rho)}. \tag{2}$$

The low type retailer orders his desired allocation because it expects to receive this allocation. The high type's order is the solution to $x_h^* = r_h^1(X^*)$, which is the optimal order assuming capacity binds only when there are two high types. Before this candidate can be proclaimed a viable equilibrium, a pair of conditions must be checked. First, it must be confirmed that the expectations indeed hold, i.e., $2x_h^* > K$ and $x_l^* + x_h^* \leq K$. Second, it must be confirmed that the high type retailers are indeed choosing a globally optimal order, i.e., $x_h^* = r_h(X^*)$. If both those tests are passed, then the candidate is indeed in equilibrium.

Note that expectations place bounds on the range of capacity values over which a Type 1 equilibrium is viable. Specifically, for such an equilibrium to exist, it must be that $2a_h > K > K_1$ where:

$$K_1 = a_l + a_h + \frac{(1-\alpha)\rho}{2(4-3\rho)}.$$

It is straightforward to show that K_1 is increasing in ρ and goes to $2a_h$ as ρ goes to one. Thus the range of capacity values over which a Type 1 equilibrium is viable shrinks and eventually disappears as the probability of a high type realization increases. Further, for positive values of ρ, $K_1 > a_l + a_h$; the equilibrium can collapse when a low type anticipates capacity binding even though in truth there is always sufficient capacity given that one retailer is a low type.

In a Type 2 symmetric equilibrium capacity binds whenever there is a high type retailer. Let $X^{**} = \{x_l^{**}, x_h^{**}\}$ be this equilibrium. Given the expectations, it must hold that $2x_l^{**} \leq K$, and $x_l^{**} + x_h^{**} > K$. The candidate equilibrium is found from the solution to the following equations

$$x_l^{**} = r_l^1\{X^{**}\}, \quad x_h^{**} = r_h^2\{X^{**}\},$$

which is

$$x_l^{**} = \frac{2(\alpha - w) + \rho(1 - \alpha - 2(K + \alpha - w)) + \rho^2(K + \alpha - w)}{4(1-\rho)^2},$$

$$x_h^{**} = \frac{4(1-w) + 2(\alpha - w) - 4K + \rho(2K - 3(1 + \alpha - 2w))}{4(1-\rho)^2} + \frac{\rho^2(K + \alpha - w)}{4(1-\rho)^2}. \tag{3}$$

As before, this candidate must pass two tests. First, the expectations must indeed hold and second, each retailer must indeed choose a globally optimal order, $x_t^{**} = r_t(X^{**})$.

Again expectations impose limits on capacity, $\overline{K}_2 > K > \underline{K}_2$ where

$$\overline{K}_2 = a_l + a_h + \frac{(1-\alpha)\rho}{2(2-\rho)},$$

$$\underline{K}_2 = 2a_l + \frac{(1-\alpha)\rho}{2 - \rho - \rho(1-\rho)}.$$

Clearly $\underline{K}_2 > 2a_l$. Previously we had said that no equilibrium could exist if capacity were insufficient to satisfy two low types. Now we see that either type of equilibria collapses at a higher capacity in a nontrivial problem (i.e., if $\rho > 0$). In addition, it is simple to show that both bounds go to $2a_h$ as ρ goes to one. As with Type 1 equilibria, the range over which Type 2 equilibria are viable collapses as the chance of a high type retailer increases. Finally note that $\overline{K}_2 > K_1$ for all admissible values of ρ. Thus for any set of model parameter values there will exist a range of capacity values such that both types of equilibria are viable.

Although two possible symmetric equilibria have been identified, there is no assurance that either type exists. In fact, there is no assurance that any pure strategy Nash equilibrium exists. The standard proof to demonstrate existence of pure strategy Nash equilibria requires convex reaction correspondences [6]. This requirement is satisfied when payoff functions are concave but the previous section has shown that the payoff functions in this game are not necessarily concave so the reaction correspondences are not necessarily convex. Of course, it is possible that either type of symmetric equilibrium exists. Further, it is even possible that both symmetric equilibria exist. Hence, there are two reasons why play in this game can be quite unpredictable. First, there may be no pure strategy Nash equilibrium, so no order corresponds to an equilibrium order. Second, there may be multiple equilibria, so each player will be uncertain whether they expect the same equilibrium as the other player.

While it can be difficult to predict play in this game, we have the following result regarding the supplier's influence on retailer behavior in either of the two possible symmetric equilibria. The proof is straightforward and therefore omitted.

Proposition 6. *With either symmetric Nash equilibrium, the supplier's expected demand is decreasing in her capacity.*

As observed in Cachon and Lariviere [5], a supplier facing the prospect of a flood of orders should not necessarily scramble to build more capacity because building additional capacity in fact lowers her demand.

4. Proportional allocation

When the supplier implements proportional allocation a retailer ordering x is allocated $a(x, y)$ when the other retailer orders y, where

$$a(x, y) = \min\left\{x, \frac{x}{x+y}K\right\}.$$

Like linear allocation, proportional allocation is conceptually simple and easy to implement. Unlike linear allocation, proportional allocation is commonly used in practice. We conjecture that several factors explain the prevalence of proportional allocation. First, it may be the most intuitive allocation scheme. It is probably the first and only algorithm most people think of when they need to determine an allocation. Second, proportional allocation seems equitable possibly because it never gives any retailer a zero allocation. (Recall, a retailer may receive nothing under linear allocation). However, as already mentioned, proportional allocation does not maximize the sum of retailers' profits in our model if the retailers were to order their desired quantities. Therefore, we study proportional allocation only because of its prevalence in practice. Surprisingly, this is the first formal analysis of a game in which proportional allocation is implemented.

4.1. *Reaction correspondence*

Define $r_t(Y)$ as a type t retailer's optimal order if he expects that Y is the other retailer's strategy. As with linear allocation, evaluation of $r_t(Y)$ is considered only for reasonable expectation, i.e., $y_t \geq a_t$ and $y_l \leq y_h$. (The proof is analogous to that of Lemma 1).

As with linear allocation, when $K - y_h \geq a_t$, retailer i can order his desired allocation a_t and be assured of receiving it. At the other extreme, when $K \leq a_t$, retailer i should submit an order for an infinite quantity because he would be happy receiving all of capacity. The remainder of this section assumes $a_t < K < a_t + y_h$, so retailer i expects that capacity might bind but does not want all of it.

The first step in the search for an optimal order is to identify a closed interval in which all profit maximizing orders must exist.

Theorem 7. *All profit maximizing orders lie in the interval* $[\underline{r}_t, \overline{r}_t]$, *where* $\overline{r}_t = r_t^*(y_h)$, $\underline{r}_t = r_t^*(y_l)$ *and*

$$r_t^*(y) = y\frac{z_t - w}{2K - (z_t - w)}.$$

Retailer i's profit function is not necessarily unimodal over the interval $[\underline{r}_t, \overline{r}_t]$. However, a numerical search quickly reveals the optimal order quantity in the interval. To assist in the search, it is possible to identify strictly concave sub-intervals.

Lemma 8. *Retailer i's profit function is strictly concave over the intervals* $[\underline{r}_t, \min\{K - y_l, \hat{r}\}]$ *and* $[\max\{\underline{r}_t, K - y_l\}, \hat{r}]$, *where*

$$\hat{r} = y_l\frac{K + (\alpha - w)}{2K - (\alpha - w)}.$$

With linear allocation we identified closed form local optima and then searched over that small set to find the set of globally optimal orders. In principle, an analogous procedure could be applied with proportional allocation. However, as demonstrated by the computer output of a common algebra software package, the closed form local optima are extremely cumbersome. Hence, we choose to implement a numerical search over a closed interval that is known to contain the globally optimal orders.

4.2. *Equilibria*

As with linear allocation, when $K \geq 2a_h$, every retailer can order his desired quantity and receive it, so that is the unique equilibrium. When $K < \max\{a_h, 2a_l\}$, some retailer always has an incentive to raise his order and no equilibrium exists.

As with linear allocation, two types of symmetric equilibria are possible with proportional allocation. Recall, in a Type 1 equilibrium only the high types inflate their orders. With proportional allocation it is possible to determine explicitly a candidate equilibrium for this case.

Theorem 9. *If there exists a Type 1 pure strategy equilibrium with proportional allocation (only high type retailers order more than their desired allocation),then it is symmetric and the high type retailers order*

$$\hat{x}_h = \hat{y}_h = \frac{(1-w)}{4}\left(1 + \sqrt{1 + \frac{2\rho K(1-w-K)}{(1-\rho)(1-w)^2}}\right).$$

The previous theorem presents a candidate equilibrium. To in fact be an equilibrium, it must hold that $\hat{x}_h + a_l \leq K$, so that each retailer indeed has the correct expectation that capacity binds only when there are two high type retailers. It must also hold that each retailer chooses a globally optimal order.

With a Type 2 symmetric equilibrium all retailers inflate their orders. However, the solution even to a candidate equilibrium is quite cumbersome. Hence, we propose a search for this equilibrium via iterative application of the reaction correspondences. (This search may also find asymmetric equilibria, if they exist). The search proceeds with the following steps:

(1) Choose an initial vector of orders, $\{\{x_l, x_h\}, \{y_l, y_h\}\}$. Set the variable **count** to zero.
(2) Store in memory the current vector of orders. Increment **count** by one.
(3) Evaluate retailer i's best response orders for each type (low and high) and update the vector of orders.
(4) Evaluate retailer j's best response orders for each type (given the updated vector of orders from the previous step) and update the vector of orders.
(5) Compare the current vector of orders with the one stored in Step 2. If the vectors are the same (within a tight tolerance), stop; an equilibrium has been found. If they are not the same and the maximum retailer order exceeds a large threshold (say one million times a_h), or if **count** is greater than a large constant (say 1000), stop; assume the procedure will not find an equilibrium. Otherwise, go to Step 2.

The above procedure should be run for several initial vectors. We suggest the vector of truthful orders and the three order vectors corresponding to the three possible candidate equilibria with linear allocation, i.e., (2), (3) and the asymmetric equilibrium candidate discussed in Appendix B. In Step 5, the "exit without finding an equilibrium" conditions are chosen because (1) it is assumed that extremely large orders just create a perpetual cycle of ever increasing orders and (2) an equilibrium probably doesn't exist if there is no convergence after a large number of iterations.

Even though it is not possible to *a priori* evaluate equilibria with proportional allocation, it is possible to show that, as with linear allocation, the supplier's expected sales will decline in her capacity. Again, the poor supplier faces the paradox that her potential sales are highest when she is unable to completely serve the market.

Theorem 10. *With any equilibrium under proportional allocation, the supplier's expected sales are declining in her capacity.*

4.3. *Behavior without an equilibrium*

Suppose the supplier implements proportional allocation but there is no equilibrium in order quantities, i.e., any order can be justified so rampant order inflation can be expected to occur. Hence, it is very likely that orders will indeed exceed capacity, which means that the supplier's sales will probably equal K and her profit will equal wK. Further, with runaway order inflation, it can be expected that the allocation of inventory between the two retailers will not reflect their true needs. For example, it is possible that even though one retailer orders substantially more than K, the other retailer might submit a significantly larger order, giving the first retailer nearly a zero allocation. Hence, it is possible that a high type retailer receives nearly a zero allocation while a low type retailer receives nearly all of capacity.

To provide some indication of the supply chain's performance when there is no equilibrium, we assume that a retailer's expected profits, $\underline{\pi}_t$, equals his profits when his allocation is uniformly distributed on the interval $[0, K]$,

$$\underline{\pi}_t = \frac{1}{K}\int_0^K \pi_t(a)\,\mathrm{d}a = a_t K - K^2/3.$$

In other words, a retailer assumes that no matter what order he submits, it is equally likely that he will be stuck with all of capacity (because the other retailer might order significantly less) or that he will be left with no capacity (because the other retailer might order substantially more). The above could be considered a lower bound on a retailer's profits since in practice retailers would probably restrain their level of order inflation somewhat (for reasons that we don't explicitly model), thereby reducing the chance that a retailer received either a very small or a very large allocation.

Overall, let $\underline{\Pi}$ be an estimate for supply chain profits when there is no equilibrium with proportional allocation,

$$\underline{\Pi} = wK + 2((1-\rho)\underline{\pi}_l + \rho\underline{\pi}_h). \tag{4}$$

5. Uniform allocation

With uniform allocation the supplier evenly divides its capacity between the two retailers. If this amount exceeds one retailer's order, then the difference is allocated to the second retailer. Of course, that second retailer will not be allocated more than his order either. Formally, a retailer ordering x is allocated $a(x,y)$ when the other retailer orders y, where

$$a(x,y) = \begin{cases} x & x+y \le K, \\ \min\{x, K/2\} & x+y \ge K,\ x \le y, \\ \min\{x, \max\{K/2, K-y\}\} & x+y \ge K,\ x > y. \end{cases}$$

With uniform allocation there is no incentive for either retailer to order any more than his desired allocation. To explain, suppose retailer i orders $x_t > a_t$. If capacity does not bind, retailer i is better off ordering a_t. If capacity binds and $x_t < y$, then retailer i receives $\min\{x_t, K/2\}$. If $a_t < K/2$, then the retailer would be better off ordering a_t. If $a_t > K/2$, then the retailer is indifferent between ordering x_t and a_t. Finally, if capacity binds and $x_t > y$, then retailer i either receives more than desired (so ordering a_t would be preferred) or the retailer receives less than a_t but would receive that amount for any order a_t or greater (so again ordering a_t is optimal). To summarize, with uniform allocation the retailer can only raise his allocation in situations for which he does not want to increase his allocation. Ordering one's desired amount is an extremely robust strategy; it is optimal no matter the value of K or the strategy used by the other retailer. It follows immediately that each retailer ordering their desired allocation is the unique Nash equilibrium with uniform allocation. (See Sprumont [8], for other properties of uniform allocation).

It is easy to show that when $K/2 > a_l$, the low type retailers always receive their desired allocation with uniform allocation. Further, when $K/2 < a_h$, the high type retailers always receive less than their desired allocation. Hence, it could be said that uniform allocation favors the smaller retailers.

Interestingly, assuming the retailers order their desired allocation, uniform allocation does not maximize the retailers' profits. As already mentioned, linear allocation maximizes their profits in that condition. (But remember, that the retailers will not actually order their desired quantities under linear allocation because that scheme harbors an incentive to inflate orders). The problem with uniform allocation is that it gives too much to the low type retailer when there is one low type retailer, one high type retailer and $a_l < K/2 < a_h$. In that setting the low type retailer receives his desired allocation (so his marginal value for additional stock is zero), whereas the high type retailer's marginal value for additional stock is positive. The supply chain could enjoy higher profits by allocating more capacity to the high type retailer.

6. Numerical study

A numerical study was conducted to gain further insight into the allocation game with either linear, proportional or uniform allocation. A set of 125 problems was constructed from all combinations of the following parameters

$$\alpha \in \{0.1, 0.3, 0.5, 0.7, 0.9\},$$
$$w \in \{0.05, 0.15, 0.25, 0.35, 0.45, 0.55, 0.65, 0.75, 0.85\},$$
$$\rho \in \{0.1, 0.3, 0.5, 0.7, 0.9\},$$

where only combinations with $\alpha > w$ are included (so that the low type retailers desire a positive amount). Recall that the game is interesting only for $K \in (K_-, K_+)$, where $K_- = \max\{2a_l, a_h\}$ and $K_+ = 2a_h$. At any capacity above that interval, the retailers order their desired quantities while for any capacity below the interval there is no equilibrium. Therefore, for each problem a set of 99 scenarios are constructed where capacity in the kth scenario is

$$K_- + \frac{k}{100}(K_+ - K_-).$$

To facilitate comparisons across problems, define the *capacity index* of the kth scenario to be $k/100$. A capacity index of 0.01 thus represents extremely tight capacity while a capacity index of 0.99 represents only slightly binding capacity.

For each scenario, we searched for both linear and proportional allocation equilibria. With linear allocation we considered the Type 1 candidate equilibrium, (2), the Type 2 candidate equilibrium, (3), and the asymmetric candidate equilibrium discussed in the Appendix. With proportional allocation we considered the Type 1 candidate equilibrium specified in Theorem 9. We also searched for other equilibria, using the truthful orders and the linear allocation candidate equilibria as initial vectors, as described in Section 4.2. Finally, for each scenario we evaluated expected profits under uniform allocation, and the lower bound estimate of profits when there is no equilibrium with proportional allocation, $\underline{\Pi}$.

6.1. Observations

This numerical study offers several observations. To begin, our focus on symmetric equilibria is justified.

- No asymmetric equilibrium was found under either linear or proportional allocation.

While we cannot claim that asymmetric equilibria never exist, it appears that they are, at the very least, uncommon. We did find symmetric equilibria, and sometimes multiple equilibria in the same scenario.

- There can be no equilibria, a unique equilibrium, or multiple equilibria in the capacity allocation game with either linear or proportional allocation.

Figure 1 demonstrates the above observation for one problem. In this problem $\alpha = 0.7$, $w = 0.05$, and $\rho = 0.5$. There exists no equilibrium under either allocation scheme when capacity is tight (a capacity of 0.779 or lower, which corresponds to a capacity index of 0.42 or lower). Also, as discussed in Section 3.2, for some intermediate capacities both equilibrium types exist. The figure also suggests that the lack of an equilibrium is more common (i.e., occurs for a wider range of capacity indexes) than multiple equilibria, a pattern that is also observed in all the other problems. Note that in this problem $a_l = 0.325$ and $a_h = 0.475$. If retailers never inflated their orders, the lowest realized total orders the supplier would receive would be 0.65, the mean total orders would be 0.8, and the maximum total orders would be 0.95. Under either proportional or linear allocation, the supplier may observe expected orders substantially above these levels in equilibrium. Finally, consistent with all the other problems, proportional allocation induced more order inflation than linear allocation in the Type 2 equilibria, but less order inflation in the Type 1 equilibria.

- As capacity becomes more restrictive, it is more likely that there does not exist an equilibrium.

This observation is seen in Fig. 1, but it is demonstrated more generally in Table 2. That table indicates the percentage of scenarios across the 125 problems in which there exists an equilibrium, and it is apparent that equilibria are less likely as capacity is reduced. This finding is intuitive; as capacity becomes more restrictive each retailer's expects to receive a smaller fraction of his order, so retailers fall into a cycle of continually larger orders in an effort to secure some capacity.

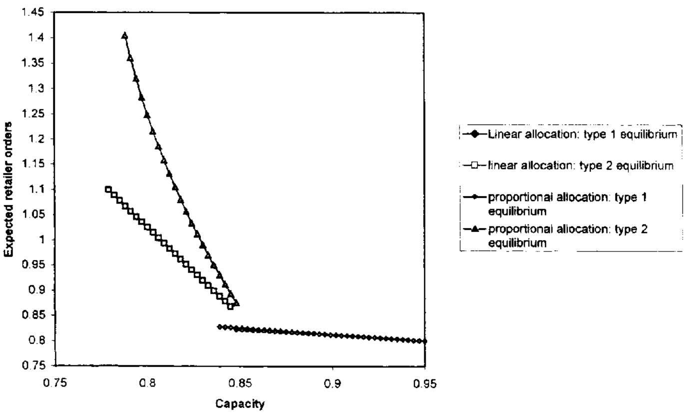

Fig. 1. Expected retailer orders ($\alpha = 0.70, w = 0.05, \rho = 0.50$).

Table 3 presents a comparison between the order inflating allocation mechanisms and uniform allocation. (To conserve space, a subset of the capacity indices is displayed). Average supply chain profits are slightly higher with either proportional or linear allocation relative to uniform allocation. Table 4 indicates that the supplier is better off with either proportional or linear allocation, while Table 5 reveals that the retailers are generally better off with uniform allocation.

- When there exists an equilibrium with either proportional or linear allocation, the supply chain is slightly better off on average than with uniform allocation.

Table 6 presents data that compares proportional and uniform allocation when there does not exist an equilibrium with proportional allocation. We assume that expected supply chain profits with proportional allocation equal $\underline{\Pi}$, as discussed in Section 4.3. The table indicates that the retailers are significantly worse off with

Table 2. Existence of equilibria

	Percentage of scenarios in which each of the following equilibria types exist			
	Linear allocation		*Proportional allocation*	
Capacity index	*Type 1 (%)*	*Type 2 (%)*	*Type 1 (%)*	*Type 2 (%)*
0.00–0.04	0	0	0	0
0.05–0.09	0	6	0	5
0.10–0.14	0	10	0	9
0.15–0.19	3	17	0	9
0.20–0.24	4	20	0	10
0.25–0.29	7	26	4	23
0.30–0.34	10	30	5	23
0.35–0.39	16	28	7	22
0.40–0.44	19	33	10	21
0.45–0.49	23	38	15	30
0.50–0.54	37	30	23	25
0.55–0.59	50	19	37	18
0.60–0.64	61	16	49	9
0.65–0.69	69	14	65	6
0.70–0.74	79	12	71	3
0.75–0.79	84	4	82	2
0.80–0.84	88	1	87	0
0.85–0.89	98	2	95	0
0.90–0.94	100	2	100	0
0.95–0.99	100	0	100	0

Table 3. Comparison of proportional, linear and uniform allocation equilibria for the supply chain

		The change in supply chain profits between proportional or linear allocation and uniform allocation, as a percentage of supply chain profits with uniform allocation			
		Proportional allocation equilibria		*Linear allocation equilibria*	
	Capacity index	*Type 1 (%)*	*Type 2 (%)*	*Type 1 (%)*	*Type 2 (%)*
Minimum of the observations	0.05		1.1		
	0.25	0.4	−2.3	0.1	−1.4
	0.50	0.0	−2.2	−1.6	−4.1
	0.75	0.0		−2.4	−1.4
	0.95	0.0		0.0	
Average of the observations	0.05		8.1		
	0.25	0.4	0.3	0.5	0.7
	0.50	0.5	0.2	1.4	0.5
	0.75	0.5		0.6	1.1
	0.95	0.2		0.2	
Maximum of the observations	0.05		24.8		
	0.25	0.5	2.1	0.8	4.2
	0.50	1.7	2.0	7.0	7.7
	0.75	3.0		3.8	4.5
	0.95	0.9		1.0	

Table 4. Comparison of proportional, linear and uniform allocation equilibria for the supplier

		The change in supplier profits between proportional or linear allocation and uniform allocation, as a percentage of supplier profits with uniform allocation			
		Proportional allocation equilibria		*Linear allocation equilibria*	
	Capacity index	*Type 1 (%)*	*Type 2 (%)*	*Type 1 (%)*	*Type 2 (%)*
Minimum of the observations	0.05		7.3		
	0.25	0.5	0.3	0.7	0.3
	0.50	0.2	0.0	0.2	0.0
	0.75	0.0		0.0	1.3
	0.95	0.0		0.0	
Average of the observations	0.05		13.2		
	0.25	0.6	1.6	1.0	2.0
	0.50	1.1	1.1	3.5	1.9
	0.75	1.2		1.6	3.8
	0.95	0.4		0.4	
Maximum of the observations	0.05		28.0		
	0.25	0.7	3.6	1.3	6.7
	0.50	2.6	4.7	10.3	8.6
	0.75	3.7		6.3	7.6
	0.95	1.0		1.1	

proportional allocation relative to uniform allocation, whereas the supplier is significantly better off. Overall, the supply chain is worse off with the order inflation created by proportional allocation. There is an explanation for these data. When there is no equilibrium with proportional allocation the retailer orders are likely to exceed capacity. With uniform allocation the retailers order their desired quantities, which may very well be less than the supplier's capacity. Hence, the supplier sells more on average with order inflation than with truthful orders, so the supplier prefers the rampant order inflation. On the other hand, the allocation of the capacity with the order inflation is quite random, thereby leading to an inferior allocation of stock across the retailers relative to uniform allocation. For example, with rampant order inflation it is possible that the retailer with the highest need does not receive the highest allocation. That clearly inefficient allocation never occurs with uniform allocation. Therefore, order inflation lowers the retailers' expected profits.

- When there is no equilibrium with proportional allocation, the supply chain and the retailers are better off with the stability of uniform allocation.

Table 7 extends the previous intuition. Rampant order inflation benefits the supplier and hurts the retailers, so it will tend to benefit the supply chain when the supplier's profits represent the lions' share of total supply chain profits. That will occur when the wholesale price is high. When the wholesale price is low, the retailers' profits represent the majority of the supply chain profit's. So we would expect that rampant order inflation would lower supply chain profits when the wholesale price is low, as is seen in Table 7.

- Uniform allocation is most valuable to the supply chain when the wholesale price is low. Alternatively, with a high wholesale price the supply chain is better off with proportional allocation.

7. Discussion

Our model is clearly a stylized version of existing supply chains. In particular, we have assumed only two demand states and two retailers. Nevertheless, we feel that the qualitative results are likely to apply in more general settings. For example, it is possible to confirm that uniform allocation induces truth-telling no matter the number of retailers, no matter the number of demand states and no matter the distribution function over demand states. Similarly, both linear and proportional allocation

Table 5. Comparison of proportional, linear and uniform allocation equilibria for the retailers

		The change in retailer profits between proportional or linear allocation and uniform allocation, as a percentage of retailer profits with uniform allocation			
		Proportional allocation equilibria		*Linear allocation equilibria*	
	Capacity index	*Type 1 (%)*	*Type 2 (%)*	*Type 1 (%)*	*Type 2 (%)*
Minimum of the observations	0.05		0.5		
	0.25	0.0	−4.8	0.0	−5.6
	0.50	−0.1	−4.3	−2.7	−5.2
	0.75	−0.4		−3.2	−2.4
	0.95	−0.1		−0.1	
Average of the observations	0.05		1.8		
	0.25	0.0	−0.9	0.0	−0.9
	0.50	0.0	−0.6	−0.5	−1.1
	0.75	−0.1		−0.2	−0.9
	0.95	0.0		−0.0	
Maximum of the observations	0.05		5.3		
	0.25	0.0	0.0	0.0	0.0
	0.50	0.0	0.0	0.0	0.0
	0.75	0.0		0.0	−0.1
	0.95	0.0		0.0	

induce order inflation in very general settings. Further, all supply chains have the potentially conflicting objectives of attempting to ensure high capacity utilization (so as to increase the supplier's profits) while also ensuring a reasonable allocation of that capacity among the retailers (to maximize their profits).

Table 6. Comparison of proportional allocation and uniform allocation when there is no proportional allocation equilibrium

	Capacity index	*Average change in profits as a percentage of uniform allocation profits (%)*
Total chain	0.05	−1
	0.25	−4
	0.50	−3
	0.75	−11
Retailers	0.05	−19
	0.25	−16
	0.50	−14
	0.75	−23
Supplier	0.05	17
	0.25	16
	0.50	11
	0.75	4

Table 7. Impact of no proportional allocation equilibrium on supply chain profits

	Average change in supply chain profits as a percentage of uniform allocation profits (%)				
	Wholesale price				
Capacity index	*0.05*	*0.25*	*0.45*	*0.65*	*0.85*
0.05	−16	−1	9	16	14
0.25	−22	−2	5	11	11
0.50	−12	−4	2	9	11
0.75	−21	−13	−6	−1	3

The primary challenge to increasing the number of demand states and retailers is computational, not conceptual. With either of those extensions it is difficult to obtain closed form solutions for candidate equilibria. A better approach to find equilibria would be to merely search for them numerically. As we do with proportional allocation, it is not difficult to place bounds on the optimal strategies, so a numerical search should not be computationally prohibitive.

There are other potentially interesting extensions to this line of research. In our model the supplier is only concerned with increasing her sales, because we have assumed that capacity is fixed before the retailers submit their orders. Therefore, the only potentially destructive element of order inflation is that it might lead to a poor allocation of capacity among the retailers. Now suppose the supplier would like to use the retailers' orders to gain some information about future demand. In that setting order inflation creates a second problem: the supplier obtains little information about demand if the retailers submit arbitrarily large orders. Overall supply chain performance could deteriorate without an exchange of credible information.

8. Conclusion

The paper studies the capacity allocation game in which a single capacity constrained supplier sells a single good to two retailers over a single period. A retailer knows whether his demand is "high" or "low" but does not know the other retailer's demand. When the sum of retailer orders exceeds capacity the supplier allocates inventory using either linear, proportional or uniform allocation. We demonstrate that the retailers have an incentive to inflate their orders with either linear or proportional allocation, but not with uniform allocation.

Behavior in the capacity allocation game is complex with either of the order inflating mechanisms. Each retailer's profit function need not be unimodal in his order quantity, so the set of profit maximizing order quantities is not necessarily convex. As a result, pure strategy Nash equilibria may not exist. Indeed, in a numerical study we often found no equilibrium, especially as capacity becomes more restrictive.

In this model the supply chain must balance two objectives: (1) increase the supplier's profits by increasing the supplier's capacity utilization; and (2) increase the retailer's profits by ensuring that the allocation of capacity closely matches the retailers' true needs. Uniform allocation suppresses order inflation, so it performs well with respect to the second goal, but may perform poorly with respect to the first goal. Inflation-inducing allocation schemes perform the first goal well, and also the second goal reasonably well when the order inflation is moderate and orderly, i.e., when there exists an equilibrium. However, when there does not exist an equilibrium those mechanisms perform the second goal poorly. Therefore, if the first goal is most important (high supplier profits), then an order-inflation allocation mechanism should be used. But if the second goal is most important (high retailer profits) then the uniform allocation rule is wiser.

To summarize, we demonstrate some allocation mechanisms always induce order inflation whenever capacity might be restrictive, whereas other mechanisms always induce retailers to order only their ideal allocation. Whether order inflation helps or harms a supply chain depends on how profits are distributed within the supply chain. Encouraging order inflation increases the supplier's profits but reduces the retailers' profits. The damage to the retailer's profits is particularly severe when the order inflation is rampant, i.e., when there is no equilibrium. That is most likely when capacity is restrictive. Hence, in those situations the supply chain is better off by implementing a truth-inducing mechanism, like uniform allocation, thereby eliminating any temptation for the retailers to game the system.

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Appendices

Appendix A Proofs

Proof of Lemma 1. A retailer never orders less than his optimal allocation because he never receives more than his allocation. (So ordering less than a_t would ensure that the retailer always received less than a_t). The latter result is demonstrated for retailer i, and then by symmetry it holds for retailer j. Suppose retailer i expects retailer j to order either y_h or y_l, where $y_h \geq y_l$. However, retailer i does not necessarily expect that retailer j will order y_h when it is a high type. Hence, either retailer i believes retailer j will order y_h with probability ρ or probability $1-\rho$. Let x^* be retailer i's optimal order when he is a high type. Certainly $x^* \geq a_h$. From the definition of x^*, for $x \geq x^*$,

$$\Pi_h(x, Y) - \Pi_h(x^*, Y) \leq 0.$$

Since

$$\frac{\partial \Pi_h}{\partial x} \geq \frac{\partial \Pi_l}{\partial x},$$

it follows that for $x \geq x^*$,

$$0 \geq \Pi_h(x, Y) - \Pi_h(x^*, Y) \geq \Pi_l(x, Y) - \Pi_l(x^*, Y).$$

But that means that $\Pi_l(x^*, Y) \geq \Pi_l(x, Y)$, so retailer i's optimal order as a low type must be no greater than his optimal order as a high type. ■

Proof of Theorem 2. The cases with unrestrictive capacity or very tight capacity have already been argued. Since profits are strictly concave in each of the first four cases in (1), the solutions to the first order conditions are clearly candidates to maximize profits. The first order condition solution to the fifth case is a_t, which cannot be optimal because capacity is restrictive when retailer j is a high type. It remains to show that it can never be optimal to order one of the interval boundaries: $K - y_l$; $y_l - K$; $y_h - K$; $y_l + K$; $y_h + K$. For the first three, retailer i is assured to receive less than a_t, so none of them can be optimal. For the last one, retailer i always receives K, which is always more than he needs. Only $y_l + K$ remains. The marginal incentive to raise his order up to $y_l + K$ is

$$\frac{(1-\rho)(z_t - w - 2K)}{2} + \frac{\rho(z_t - w - y_l + y_h - 2K)}{2}.$$

If ordering $y_l + K$ is optimal, then retailer i must have a marginal incentive to raise his order to $y_l + K$, so the above must be positive. Since $K > a_h = (1-w)/2$, the first term above is negative, which means the second term must be positive. But retailer i's marginal incentive to raise his order above $y_l + K$ is

$$\frac{\rho(z_t - w - y_l + y_h - 2K)}{2},$$

which is positive. So if retailer i is willing to order at least $y_l + K$, then he is willing to order even more. Thus, ordering $y_l + K$ cannot be optimal. ■

Proof of Lemma 3. Suppose $x_h \geq K + y_l$ is an equilibrium, so retailer j receives a zero allocation when he is a low type. Retailer i's optimal order when he is a high type is $1 - w - K + y_h$, which is greater than y_h (since $K < 2a_h = 1 - w$). This means that retailer j receives $K - a_h$ when he is a high type. For this to be an equilibrium, a high type retailer j can have no incentive to raise marginally his order

$$\frac{\rho}{2}(1 - w - 2a(y_h, x_h)) + (1 - \rho)(1 - w - 2a(y_h, x_l)) \times \frac{\partial a(y_h, x_l)}{\partial y_h} = 0,$$

nor can he have an incentive to marginally lower his order

$$\rho(1 - w - 2a(y_h, x_h)) + (1 - \rho)(1 - w - 2a(y_h, x_l)) \times \frac{\partial a(y_h, x_l)}{\partial y_h} = 0.$$

But $a(y_h, x_h) = K - a_h$, so

$$1 - w - 2a(y_h, x_h) = 2(2a_h - K) > 0.$$

Therefore, both conditions cannot be satisfied, i.e., retailer j will have an incentive to either raise or lower his order. ■

Proof of Theorem 4. From Theorem 2, in equilibrium a retailer's order must be a subset of $\{r_t^1(Y), r_t^2(Y), r_t^3(Y), r_t^4\{Y\}\}$. However, Lemma 3 eliminates any $x_h \geq K + y_l$ or $y_h - K \geq x_l$ from consideration, which eliminates $r_t^3(Y)$ and $r_t^4\{Y\}$ from consideration. ■

Proof of Lemma 5. Suppose there exists a Nash equilibrium in which even a low type expects capacity will bind when it faces another low type, i.e., $x_l + y_l > K$. From Theorem 4, the Nash equilibrium must satisfy each retailer's first order condition, so the following system of equations must be satisfied:

$$x_l = r_l^2(Y);\ x_h = r_h^2(Y);\ y_l = r_l^2(X);\ y_h = r_h^2(X).$$

But there is no solution to that system, hence there is no equilibrium. ■

Proof of Theorem 7. If retailer i knew that retailer j would order y, he would order so that he receives his desired allocation, a_t, exactly. Let $r_t^*(y)$ be this order,

$$a(r_t^*(y), y) = a_t,$$

or

$$r_t^*(y) = y\frac{z_t - w}{2K - (z_t - w)}.$$

Since retailer j orders no more than y_h, retailer i should not order more than $\bar{r}_t = r_t^*(y_h)$, because otherwise he surely will receive more than he desires. Similarly, since y_l is the least retailer j will order, retailer i should not order less than $\underline{r}_t = r_t^*(y_l)$, because otherwise he surely will receive less than he desires no matter retailer j's order. Therefore, all profit maximizing orders must be in the interval $[\underline{r}_t, \bar{r}_t]$. ■

Proof of Lemma 8. Take derivatives of the profit function:

$$\frac{\partial \Pi_t(x, Y)}{\partial x} = \begin{cases} (1-\rho)(z_t - w - 2x) + \rho\partial\pi_t(a(x, y_h))/\partial x \\ \quad x < K - y_l \\ (1-\rho)\partial\pi_t(a(x, y_l))/\partial x + \rho\partial\pi_t(a(x, y_h))/\partial x \\ \quad K - y_l \leq x, \end{cases}$$

where for $x + y \geq K$,

$$\frac{\partial \pi_t(a(x, y))}{\partial x} = \left(z_t - w - \frac{2xK}{x + y}\right)\frac{yK}{(x + y)^2};$$

and

$$\frac{\partial^2 \Pi_t(x, Y)}{\partial x^2} = \begin{cases} -2(1-\rho) + \rho\partial^2\pi_t(a(x, y_h))/\partial x^2 & x < K - y_l \\ (1-\rho)\partial^2\pi_t(a(x, y_l))/\partial x^2 & \\ \quad + \rho\partial^2\pi_t(a(x, y_h))/\partial x^2 & K - y_l \leq x, \end{cases}$$

where for $x + y \geq K$,

$$\frac{\partial^2 \pi_t(a(x, y))}{\partial x^2} = -2\left(z_t - w - \frac{2xK}{x + y} + \frac{yK}{x + y}\right)\frac{yK}{(x + y)^3}.$$

$\pi_t(a(x, y_h))$ is strictly concave for $x \in [\underline{r}_t, \bar{r}_t]$, since $K - y_h < \underline{r}_t$. $\pi_t(a(x, y_l))$ is strictly concave for $x \in [\underline{r}_t, K - y_l]$, so profits are strictly concave over $[\underline{r}_t, K - y_l]$. (Profits are the sum of these functions). For $x > K - y_l$, $\pi_t(a(x, y_l))$ can be a concave-convex function. From second order conditions, $\pi_t(a(x, y_l))$ is strictly convex for $x \geq \widehat{r}_t$, and strictly concave for $x \in [K - y_l, \widehat{r}_t]$, so profits are strictly concave on $[\max\{\underline{r}_t, K - y_l\}, \widehat{r}_t]$. ■

Proof of Theorem 9. Given the expectation that capacity binds only when there are two high type retailers, a high type retailer's marginal profit is

$$\frac{\partial \Pi_h}{\partial x_h} = (1 - \rho)(1 - w - 2x_h) + \rho\left(1 - w - \frac{2x_h}{x_h + y_h}K\right)\frac{y_h}{(x_h + y_h)^2}K,$$

and

$$\frac{\partial^2 \Pi_h}{\partial x_h^2} = -2(1-\rho) - \frac{2\rho y_h K}{(x_h+y_h)^3}\left((1-w) + K\frac{y_h}{x_h+y_h} - \frac{2x_h}{x_h+y_h}K\right).$$

With the optimal order it must hold that

$$1 - w - \frac{2x_h}{x_h+y_h}K > 0,$$

or

$$x_h < y_h \frac{1-w}{2K-(1-w)}, \tag{A1}$$

otherwise the retailer could increase his profits by decreasing his order. But if the above holds, the second order condition is negative, and hence profits are strictly concave. So the first order condition in that interval yields a global optimal.

Suppose an asymmetric equilibrium exists, i.e., the retailers choose different quantities when they are a high type. Let x_h and y_h be these quantities. Both must satisfy the first order conditions

$$(1-\rho)(1-w-2x_h) + \rho\left(1 - w - \frac{2x_h}{x_h+y_h}K\right) \times \frac{y_h}{(x_h+y_h)^2}K = 0,$$

$$(1-\rho)(1-w-2y_h) + \rho\left(1 - w - \frac{2y_h}{x_h+y_h}K\right) \times \frac{x_h}{(x_h+y_h)^2}K = 0.$$

But that implies that

$$-2(1-\rho) = \frac{\rho(1-w)K}{(x_h+y_h)^2},$$

which cannot be. So there does not exist an asymmetric equilibrium in which capacity binds only with the high types. The solution to the following yields the high type order in a symmetric equilibrium,

$$(1-\rho)(1-w-2x_h) + \rho(1-w-K)\frac{K}{4x_h} = 0.$$

It is easy to confirm that indeed (A1) holds in equilibrium, since $K < 1 - w$. ■

Proof of Theorem 10: From the implicit function theorem,

$$\frac{\partial r_t(y)}{\partial K} = -\frac{\partial^2 \Pi_t}{\partial x_t \partial K} \Big/ \frac{\partial^2 \Pi_t}{\partial x_t^2}.$$

With proportional allocation profits are continuous in a retailer's order. Therefore, any globally optimal order must satisfy a retailer's first order condition, and at that order the profit function must be concave. Hence, the above is negative, i.e., a marginal increase in capacity reduces a retailer's optimal order quantity (locally). Hence, an increase in capacity will reduce the retailers' order quantities in equilibrium, leading to lower expected sales for the supplier. ■

Appendix B. Asymmetric equilibria with linear allocation

Consider the possibility of an asymmetric equilibrium, $\overline{X} = \{\bar{x}_l, \bar{x}_h\}$ and $\overline{Y} = \{\bar{y}_l, \bar{y}_h\}$, where retailers expect capacity only binds when retailer i is a high type,

$$\bar{x}_h + \bar{y}_h > K, \quad \bar{x}_h + \bar{y}_l > K, \quad \bar{y}_h + \bar{x}_l \le K.$$

Given those expectations, the following must hold

$$\bar{x}_l = a_l, \quad \bar{x}_h = r_h^2\{\overline{Y}\}, \quad \bar{y}_l = r_l^1\{\overline{X}\}, \quad \bar{y}_h = r_h^1\{\overline{X}\}.$$

The solution is

$$\bar{x}_l = a_l,$$

$$\bar{x}_h = \frac{4(1-w-K) + 2(\alpha - w) - \rho(1 - 2K + 3\alpha - 4w)}{4(1-\rho)} - \frac{\rho^2(1-\alpha)}{4(1-\rho)},$$

$$\bar{y}_l = \frac{8(\alpha - w) - 2\rho(4K + 5\alpha - 2 - 3w)}{4(1-\rho)(4-3\rho)} - \frac{\rho^2(1-\alpha-6K) - \rho^3(1-\alpha)}{4(1-\rho)(4-3\rho)},$$

$$\bar{y}_h = \frac{8(1-w) - 2\rho(4K - \alpha + 4 - 3w)}{4(1-\rho)(4-3\rho)} + \frac{3\rho^2(1-\alpha+2K) - \rho^3(1-\alpha)}{4(1-\rho)(4-3\rho)}.$$

As with the other candidate equilibria, this equilibrium must be consistent with the retailers' expectations and no retailer can have a profitable deviation from the equilibrium, i.e.,

$$\bar{x}_l = r_l(\overline{Y}), \quad \bar{x}_h = r_h\{\overline{Y}\}, \quad \bar{y}_l = r_l\{\overline{X}\}, \quad \bar{y}_h = r_h\{\overline{X}\}.$$

Note that this is an asymmetric equilibrium, i.e., retailer i orders different quantities than retailer j. Since the retailer names are arbitrary, if such an equilibrium exists, then there necessarily exists another equilibrium in which the retailers' roles are reversed, i.e., capacity only binds when retailer j is a high type.

Biographies

Gérard Cachon is an Associate Professor of Operations Management at the Fuqua School of Business at Duke University. His research interests include multi-echelon inventory theory, supply chain management and game theory applications to operations management. He teaches an elective in supply chain management and the core operation class.

Martin A. Lariviere is an Assistant Professor of Business Administration at Duke University's Fuqua School of Business. His research interests include contracting in supply chains and channels of distribution, application of economics in marketing and operations management, and the management of the marketing/manufacturing interface.